

# Automorphic Inflation

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## Abstract

A framework of inflation is formulated based on symmetry groups and their associated automorphic functions. In this setting the inflaton multiplet takes values in a curved target space constructed from a continuous group  $G$  and a discrete subgroup  $\Gamma$ . The dynamics of inflationary models is essentially determined by the choice of the pair  $(G, \Gamma)$  and a function  $\Phi$  on the group  $G$  relative to  $\Gamma$ . Automorphic inflation provides a natural structure in which the shift symmetry of large field inflation arises as one of generators of  $\Gamma$ . The model of  $j$ -inflation is discussed as an example of modular inflation associated to the special linear group.

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## 1 Introduction

Symmetries have been useful as guides to the dynamics of fundamental theories for more than a century. The most dramatic examples involve continuous groups, but discrete groups have been of importance as well. The purpose of this paper is to formulate a field theoretic framework based on the combination of both types of groups as well as an associated function space. Given a continuous Lie group  $G$  and a discrete subgroup  $\Gamma$  of  $G$ , the dynamics is determined by the choice of automorphic forms  $\Phi_i$ , defined relative to the discrete group  $\Gamma$  as a function on  $G$ . These forms descend to functions  $f_i$  on the target space  $X$  of the inflaton multiplet  $\phi^I$ , thereby inducing a potential  $V(f_i(\phi^I))$ . The kinetic term of the resulting theory leads to a non-flat metric  $G_{IJ}$ , determined by the Lie algebra of the group  $G$ .

In this letter the framework of automorphic field theory is applied to inflation. This is motivated by issues that arise in models in which inflation occurs at a high energy scale. Estimates of the effects of higher dimension operators expected to appear in

the UV-completion of inflationary models suggest that such operators can make model-specific predictions unstable. This is particularly pronounced in the framework of large field inflation, where the inflaton varies over an energy range that is super-Planckian. Unless these operators have very small coefficients, higher dimension corrections will have dramatic effects on the parameters. An early discussion of these issues in the context of chaotic inflation [1] can be found in ref. [2]. A device often postulated to avoid such corrections is the existence of an inflaton shift symmetry  $\phi \mapsto \phi + s$ . Historically, the first model to incorporate this idea is natural inflation [3], but many modifications and extensions have been introduced in the intervening two decades, including models that aim at realizations of this symmetry in UV-complete theories [4, 5, 6]. This idea has received renewed attention following the possibility of an observable tensor ratio [7, 8, 9]. If a sizable portion of the CMB power spectrum comes from primordial gravitational waves the inflationary scale is quite high, not too far from that of GUT models [10].

In the context of the shift symmetry it is natural to ask whether it is part of a larger group that operates on the inflaton field space. The existence of such a group would provide a systematic framework in which different types of invariant potentials could be classified. In the present letter such a program is initiated by formulating inflation in terms of symmetry groups and their associated automorphic functions. There is a historical precedent for such a strategy in quantum field theory, where the interpretation of an inversion symmetry as a generator of an infinite group led to the concept of duality. In the present framework the shift symmetry becomes part of an infinite subgroup  $\Gamma$  of  $G$  with constraints strong enough to characterize the space of forms.

## 2 Automorphic inflation framework

The structure of automorphic inflation is characterized in essence by a potential determined in terms of automorphic forms, and a metric  $G_{IJ}$  of the kinetic term that is determined by the symmetry group.

## 2.1 Automorphic actions

In the simplest case the concept of automorphic inflation can be formulated in the context of multi-field theories defined by an action of the form

$$\mathcal{A}_{\text{aut}} = \int d^4x \sqrt{-g} \left( \frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} G_{IJ} \partial_\mu \phi^I \partial^\mu \phi^J - V(\phi^I) \right), \quad (1)$$

where  $M_{\text{Pl}} = 1/\sqrt{8\pi G}$  is the reduced Planck mass and the spacetime metric  $g_{\mu\nu}$  is taken to be of signature  $(-, +, +, +)$ . The basic building blocks include potentials  $V(\phi^I)$  induced by automorphic forms  $f$  on the inflaton field space  $X$  with coordinates  $\tau = (\tau^1, \dots, \tau^n)$  as  $V(f(\tau^I)) = \Lambda^4 F(f(\tau^I))$ . Here  $\Lambda$  is an energy scale, and  $F(\tau^I)$  is a dimensionless function of the inflaton multiplet expressed in terms of dimensionless variables  $\tau = \phi/\mu$ , where  $\mu$  is a second energy scale. The potential function  $F$  should be a real function  $F(f, \bar{f})$ , a simple class of examples given by  $F(f, \bar{f}) = (f\bar{f})^p$  for arbitrary exponents  $p$ . Inflationary models  $\mathcal{I}$  of this type are thus characterized by a number of choices that characterize the groups  $(G, \Gamma)$ , the functions  $(\Phi, F)$ , and the energy scales  $(\Lambda, \mu)$ . The space  $X$  is to be taken as a bounded region in the complex vector space of dimension  $n$ , whose structure is constrained by the nature of the inflaton  $\phi^I$ . These spaces have a curved metric  $G_{IJ}$ ,  $I, J = 1, \dots, n$ , which will be described in more detail further below, after the nature of the space  $X$  has been made more explicit.

Automorphic inflation can be formulated in a more general context for theories of the form  $P(E, V)$ , where  $E = G_{IJ} g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \bar{\phi}^J$ , and  $P$  is some general, not necessarily polynomial, function. Examples of this type include the extension of DBI inflation [11, 12] to the multi-field context.

## 2.2 Automorphic forms associated to groups

The concept of automorphic forms was introduced in the second half of the 19<sup>th</sup> century in the context of functions on the complex plane. This framework is too narrow for multi-field inflation, but the generalization of the classical notion of automorphic forms has

made the concept less precise and no standard language has emerged. In the following discussion automorphic forms are understood to be defined on higher rank matrix groups, to be distinguished from modular forms. Roughly speaking, forms of this type are highly symmetric functions on a continuous group  $G$  that are characterized by their transformation behavior with respect to certain subgroups, and by the fact that they are eigenfunctions of differential operators. In this way they can be viewed as objects reminiscent of tensors, but more rigid. When  $G$  is a semisimple Lie group with finite center there are essentially only two subgroups to consider, the discrete subgroup  $\Gamma$  and the maximal compact subgroup  $K$ . The inflaton space  $X$  is defined as the quotient space  $X = G/K$ , which inherits a Riemannian metric from  $G$ . It is therefore natural to consider eigenfunctions of those differential operators that are invariant under the group action.

In more detail, the transformation behavior of the function  $f$  on  $X$  with respect to the discrete group  $\Gamma$  is determined by an automorphy factor  $J(g, x)$  that depends both on group elements  $g$  and on elements  $x$  of  $X$ . This function  $J$  is defined by the relation  $J(gh, x) = J(g, hx)J(h, x)$  for  $g, h \in G$  and  $x \in X$ . A  $J$ -automorphic form  $f(x)$  is induced by a certain type of group function  $\Phi(g)$  as

$$f(x) = J(g, x_0)\Phi(g), \quad (2)$$

where  $x = g \cdot x_0$  and  $x_0$  is a point left invariant by the subgroup  $K$ . In order to obtain the standard transformation behavior as

$$f(\gamma x) = \epsilon(\gamma)J(\gamma, x)f(x), \quad (3)$$

where  $\gamma$  is an element of  $\Gamma$  and  $x$  is in  $X$ , the functions  $\Phi$  have to satisfy a number of constraints that are mostly concerned with the transformation behavior of  $\Phi$  with respect to different kinds of subgroups of the group  $G$ . These constraints will be illustrated more concretely in the special case of modular forms further below.

If  $G$  is a semisimple Lie group with finite center the first condition specifies the behavior of the function  $\Phi$  with respect to the action of the discrete group  $\Gamma$  on  $\Phi$ , which in

general is allowed to transform with a character  $\epsilon$  as  $\Phi(\gamma g) = \epsilon(\gamma)\Phi(g)$ , for  $\gamma \in \Gamma$ . The second constraint restricts the behavior of  $\Phi$  with respect to the action of the maximal compact subgroup  $K$  by requiring that the forms span a finite-dimensional space under this action. The third condition, structurally different in type from the transformation constraints above, generalizes the eigenvalue constraints of classical modular forms, either of holomorphic or non-holomorphic type. In the higher rank case the function  $\Phi$  is assumed to be an element of a finite dimensional eigenspace of all differential operators that are invariant with respect to the group structure. The number of generating operators of this type is finite, given by the rank of the group  $G$ . Finiteness results for automorphic forms have been established by imposing a convergence constraint that requires the existence of a constant  $C$  and an integer  $m$  such that  $|\Phi(g)| \leq C||g||^m$ . More details concerning the conceptual framework of automorphic forms can be found in the reviews of ref. [13].

The notion of automorphic inflation just outlined in terms of the group theoretic framework associates different models to each Lie group  $G$  and the subgroup  $\Gamma$  via the choice of an automorphic form  $\Phi$  and the function  $F$ . Once a symmetric bounded domain has been chosen, together with  $\Gamma$ , the space of automorphic forms is finite-dimensional [14, 15]. This implies that modulo the function  $F$  each choice  $(G, \Gamma)$  leads to a finite-dimensional theory space.

### 2.3 The kinetic term of automorphic inflation

The kinetic term of automorphic field theory is characterized by a nontrivial metric on the target space  $X = G/K$  of the inflaton multiplet. This metric is induced in terms of the adjoint representation of the Lie algebra  $\mathfrak{g}$ , defined as  $\text{ad}_V(W) = [V, W]$ , via the Cartan-Killing form  $B(X, Y) = \text{tr ad}_X \text{ad}_Y$  on  $\mathfrak{g}$  of  $G$ , which is isomorphic to the tangent space  $T_e G$  at the identity element  $e$ . The inner product on  $T_e G$  can be transported to other tangent spaces  $T_g G$  by the differential  $dL_g$  of the left translation

map  $L_g$ . The pullback via  $L_{g^{-1}}$  can be used to define the inner product on  $T_g G$  as  $\langle V, W \rangle_g = \langle dL_{g^{-1}}V, dL_{g^{-1}}W \rangle_e$ , for tangent vectors  $V, W$ . The associated metric descends to the quotient  $X = G/K$ .

The coordinate form  $G_{IJ}$  of the metric on  $X$  can be obtained explicitly from the Iwawasa decomposition  $G = NAK$  of  $G$ , defined as a refinement of the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , where  $\mathfrak{k}$  is the Lie algebra of the maximal compact subgroup  $K$ . By choosing a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  the decomposition  $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}$  leads to factors  $NAK$  of the group  $G$ , where  $N$  is nilpotent and  $A$  is abelian. The map from  $G$  to  $X$  can be made explicit by choosing the maximal compact subgroup  $K$  to be the isotropy group of a point  $x_0$  in  $X$ , leading to  $gx_0 = nakx_0 = nax_0$ .

In the remainder of this paper this framework will be illustrated in the context of modular inflation, with  $j$ -inflation as a particular example.

### 3 Modular inflation

The simplest theories of automorphic inflation can be formulated in the context of classical modular functions and forms, which are functions understood to be defined relative to arithmetic subgroups of the modular group  $\mathrm{SL}(2, \mathbb{Z})$ , such as the Hecke congruence subgroups  $\Gamma_0(N)$  of level  $N$ , the principal congruence subgroups  $\Gamma(N)$ , or other similar groups, collectively denoted by  $\Gamma_N$ . The semisimple group  $G$  in this case is  $\mathrm{SL}(2, \mathbb{R})$ , and the maximal compact group  $K = \mathrm{SO}(2, \mathbb{R})$  leads to the domain  $X$  which can be viewed as the upper halfplane  $\mathcal{H} = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ .

For modular forms  $\Phi$  defined as group functions on  $\mathrm{SL}(2, \mathbb{R})$  the first automorphy constraint again identifies a character  $\epsilon_N$ , possibly trivial, such that  $\Phi(\gamma g) = \epsilon_N(\gamma)\Phi(g)$ . The second constraint,  $K$ -finiteness, can be made more precise in the modular case because the compact subgroup is abelian, hence its irreducible representations are one-dimensional. The resulting character determines the weight  $w$  of the form via  $\Phi(k_\theta g) =$

$e^{iw\theta}\Phi(g)$ , where  $k_\theta$  is a rotation by  $\theta$ . Finally, the differential constraint simplifies as well because the rank of  $\text{SL}(2, \mathbb{R})$  is one, hence there is essentially only one invariant differential operator, the Casimir element  $\mathcal{C} = h^{ij}X_iX_j$ , where  $h^{ij}$  is the inverse of the Killing metric  $h_{ij}$ , defined with respect to a basis  $\{X_i\}$  of  $\mathfrak{g}$  as  $h_{ij} = \text{tr ad}_{X_i}\text{ad}_{X_j}$ . The differential operator image  $\Delta_{\mathcal{C}}$  of  $\mathcal{C}$  is a multiple of the Laplace-Beltrami operator  $\Delta_g = g^{-1/2}\partial_i g^{ij}\sqrt{g}\partial_j$ . The eigenform constraint thus distinguishes between holomorphic and non-holomorphic forms.

Modular forms  $f(\tau)$  on the upper halfplane, characterized by their weight  $w$ , level  $N$  and character  $\epsilon$ , are obtained from group functions  $\Phi$  on  $\text{SL}(2, \mathbb{R})$  by considering the base point  $x_0 = i = \sqrt{-1}$  of the maximal compact subgroup  $K$  as  $f(\tau) = J(g, i)\Phi(g)$  where  $\tau = g \cdot i$ . For elements  $\gamma$  in the discrete subgroup  $\Gamma_N$  of level  $N$  the automorphy factor is  $J(\gamma, \tau) = (c\tau + d)^w$ , where  $\gamma$  is an element of  $\Gamma_N$  with rows  $(a, b)$  and  $(c, d)$ , and the transformation behavior is given by

$$f(\gamma\tau) = \epsilon_N(\gamma)(c\tau + d)^w f(\tau), \quad \gamma \in \Gamma_N, \quad (4)$$

where  $\det(\gamma) = 1$ , and  $\gamma\tau = (a\tau + b)/(c\tau + d)$  describes the fractional transform on the upper halfplane. The shift symmetry arises in the modular case from the generator given by  $(a, b) = (1, 1)$  and  $(c, d) = (0, 1)$ , leading to the transformation that sends  $\tau$  to  $\tau + 1$ . This shift symmetry also embeds into higher rank groups  $G$  via the embedding of  $\text{SL}(2, \mathbb{R})$  into  $G$ .

The inflaton doublet  $\phi = (\phi^1, \phi^2)$  is parametrized by  $\phi = \mu\tau$  and the target space is equipped with the hyperbolic metric  $ds^2 = d\tau d\bar{\tau}/(\text{Im } \tau)^2$ , i.e. the metric of the kinetic term is conformally flat with  $G_{IJ} = (\mu/\phi^2)^2 \delta_{IJ}$ .

Given a continuous group  $G$ , such as  $\text{SL}(2, \mathbb{R})$ , model building proceeds by first choosing a discrete subgroup  $\Gamma$ , such as  $\Gamma_N \subset \text{SL}(2, \mathbb{Z})$ . This determines the level structure of the inflationary model. For a given pair  $(G, \Gamma)$ , different modular functions can be obtained by considering quotients  $f(\tau) = g(\tau)/h(\tau)$  of modular forms  $g, h$  of the same weight with respect to the same discrete group. Such functions then induce inflationary potentials of



the type  $V(f) = \Lambda^4 F(f, \bar{f})$ . This set-up of modular inflation thus provides an extensive framework in which finite-dimensional Hilbert spaces of modular forms can be used to construct modular invariant potentials. The example of  $j$ -inflation considered below uses a particular modular function, constructed from modular forms with respect to the full modular group, but other functions with respect to the full group can be considered, as well as forms of higher level  $N > 1$ .

## 4 $j$ -Inflation

The framework of modular inflation is exemplified in the following by considering an inflaton potential determined by a function invariant with respect to the full modular group. The space of classical modular forms with respect to  $\text{SL}(2, \mathbb{Z})$  is in principle completely known since it is spanned by only two modular forms, the Eisenstein series  $E_4(\tau)$  and  $E_6(\tau)$  of weight 4 and 6 on the upper halfplane. These forms arise from the general Eisenstein series on the group  $\text{SL}(2, \mathbb{R})$  defined as  $E_s(g, f) = \sum_{\gamma} f_s(\gamma g)$ , where the sum is over a discrete quotient group,  $s$  is a free parameter,  $g$  is in  $\text{SL}(2, \mathbb{R})$ , and  $f_s(g)$  is a rescaled form of the function  $f$  with a weight determined by  $s$ . The specific structure of  $f_s$  encodes the form of the different Eisenstein series on the upper halfplane. The normalized Eisenstein series  $E_w(\tau)$  of even weight  $w$  that result from  $E_s(g, f)$  can be expressed in terms of the divisor function  $\sigma_m(n) = \sum_{d|n} d^m$  as

$$E_w(\tau) = 1 - \frac{2w}{B_w} \sum_n \sigma_{w-1}(n) e^{2\pi i n \tau}, \quad (5)$$

where  $\tau$  is in  $\mathcal{H}$  and  $B_w$  are the Bernoulli numbers, which are related via Euler's formula to the Riemann zeta function as  $2w! \zeta(w) = -(2\pi i)^w B_w$ , with  $w$  a positive even integer. The forms  $E_4$  and  $E_6$  are the unique elements of spaces of forms of weight four and six, respectively, up to normalization.

Holomorphic modular functions can be constructed from modular forms by considering quotients such that the denominator form is non-vanishing in the upper halfplane. A

venerable modular form with this property is the discriminant function  $\Delta$ , the unique cusp form of weight twelve, up to normalization, defined by  $\Delta(\tau) = \eta(\tau)^{24}$ , where  $\eta(\tau)$  is the Dedekind function  $\eta(\tau) = e^{2\pi i\tau/24} \prod_{n \geq 1} (1 - e^{2\pi i n \tau})$ . The form  $\Delta$  arises in many different physical contexts, for example the partition function of the bosonic string, or as a geometric object, but is considered here simply as a building block of modular functions. Because  $\Delta$  does not vanish on  $\mathcal{H}$ , modular functions without poles in  $\mathcal{H}$  can be obtained by considering numerators of weight twelve. One of these is  $E_4^3$ , leading to the elliptic modular function

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}, \quad (6)$$

which is perhaps the most prominent modular function, going back to Kronecker's Jugendtraum, in particular his work on the class numbers of imaginary quadratic fields [16], and Hermite's work on the quintic equation [17]. Both  $E_4$  and  $\Delta$  are modular forms with respect to the full modular group, hence  $j$  is a holomorphic function invariant under  $\text{SL}(2, \mathbb{Z})$ .

Given the function  $j$  one can consider  $j$ -inflationary models defined by potentials  $V = \Lambda^4 F(j, \bar{j})$ . The simplest cases are obtained by setting e.g.  $F(f, \bar{f}) = (f\bar{f})^p$  and in the following the focus will be on the model with  $p = 1$ . As a first step toward a phenomenological analysis it is most convenient to consider the slow-roll approximation since this allows an analytic discussion of many aspects of the model, leaving a more precise numerical analysis of the exact dynamics to a more detailed discussion. Important parameters emphasized by the WMAP and PLANCK analyses are given by the scalar spectral index  $n_s$ , defined in terms of the scalar power spectrum  $\mathcal{P}_s(k)$  as  $n_s - 1 = d \ln \mathcal{P}_s / d \ln k$ , and the tensor-to-scalar ratio  $r$ , defined via the tensor power spectrum  $\mathcal{P}_t$  as  $r = \mathcal{P}_t / \mathcal{P}_s$ . Similar to  $n_s$  one considers the tensor power spectral index defined as  $n_t = d \ln \mathcal{P}_t / d \ln k$ .

The power spectrum of the curvature perturbation can be written in multi-field inflation in terms of the number of  $e$ -foldings  $N$  as [18]

$$\mathcal{P}_s = \left( \frac{H}{2\pi} \right)^2 G^{IJ} \frac{\partial N}{\partial \phi^I} \frac{\partial N}{\partial \phi^J}, \quad (7)$$

while the tensor power spectrum was determined by Starobinsky [19] to be given by the Hubble parameter

$$\mathcal{P}_t = \frac{2}{\pi^2} \left( \frac{H}{M_{\text{Pl}}} \right)^2. \quad (8)$$

For the inflationary model based on the  $j$ -function the parameters  $n_s$  and  $r$  can be computed analytically in the slow roll approximation in terms of the dimensionless parameters  $\epsilon_I$  and  $\eta_{IJ}$ , defined as  $\epsilon_I = M_{\text{Pl}} V_{,I}/V$  and  $\eta_{IJ} = M_{\text{Pl}}^2 \nabla_I \nabla_J V/V$ , where  $\partial_I V = \partial V/\partial \phi^I$  and  $\nabla_I$  denotes the covariant derivative in field space. If the  $\epsilon_I$  are sufficiently small the universe accelerates.

In multi-field inflation the spectral indices can be obtained in the slow-roll approximation as

$$n_s = 1 - 3\epsilon_I \epsilon^I + 2 \frac{\eta_{IJ} \epsilon^I \epsilon^J}{\epsilon_K \epsilon^K} \quad (9)$$

and  $n_t = -\epsilon_I \epsilon^I$ , while the tensor-to-scalar ratio takes the form  $r = -8n_t$ . Here the indices are lowered and raised with the field space metric  $G_{IJ}$  and its inverse.

The observables  $\mathcal{P}_s(k_*)$ ,  $n_s$  and  $r$  can be expressed in the case of  $j$ -inflation in terms of the modular forms  $E_4, E_6$  as well as the quasi-modular form  $E_2$  by using the relation

$$\epsilon_I = -2\pi i^I \frac{M_{\text{Pl}}}{\mu} \left( \frac{E_6}{E_4} + (-1)^I \frac{\bar{E}_6}{\bar{E}_4} \right), \quad (10)$$

where the index  $I = 1, 2$  indicates the two components of  $\tau = \tau^1 + i\tau^2$ . The height  $\Lambda$  of the potential does not enter the scalar spectral index  $n_s$  and the tensor ratio  $r$ , leading to expressions  $n_s = n_s(\mu, E_w)$  and  $r = r(\mu, E_w)$  with  $w = 2, 4, 6$ . With these explicit results the spectral index and the tensor ratio can be evaluated in terms of the inflaton variable  $\tau_* = \phi_*/\mu$  at horizon crossing in dependence of the scale  $\mu$ . Values of the dimensionless inflaton  $\tau$  in the neighborhood of the zero  $\tau = i$  of  $E_6$  lead to  $(\mathcal{P}_s(k_*), n_s, r)$ -parameters that are consistent with the observational results reported by the PLANCK Collaboration [20]. More precisely, the spectral index  $n_s$  and a tensor-to-scalar ratio  $r$  with values below the Planck bound can be obtained from a wide range of scales for  $\mu$  around the Planck scale  $M_{\text{Pl}}$ . The scale  $\Lambda$  that results from the

scalar amplitude  $\mathcal{P}_s(k_*)$  includes the range  $(10^{-5} - 10^{-3})M_{\text{Pl}}$ . For super-Planckian  $\mu$  the number of e-foldings includes the standard interval between  $N_e = 50$  and  $N_e = 70$ , with  $n_s \cong 0.96*$ , and the tensor ratio includes the range  $0.02 \geq r \geq 10^{-4}$ , which is within reach of proposed experiments, such as CMBPol [21] and PRISM [22]. The values of the inflaton components  $\phi_*^I$ , determined by the choice of  $\mu$ , range from sub-Planckian to super-Planckian values for  $\phi_*^1$ , and generically super-Planckian values for  $\phi_*^2$ .

It is similarly possible to constrain further parameters of these models for which WMAP and PLANCK have provided limits. Among these are the running  $\alpha_s$  of the scalar spectral index, as well as the non-Gaussianity parameter  $f_{\text{NL}}$  of the bispectrum  $B(\vec{k}_1, \vec{k}_2, \vec{k}_3)$  and the parameters  $\tau_{\text{NL}}$  and  $g_{\text{NL}}$  of the trispectrum  $T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ . These parameters can be expressed in terms of the Eisenstein series as well, leading again to analytic expressions, and the results can be compared with the limits imposed by the PLANCK Collaboration [23]. These results involve higher derivative features of the potential and their more involved analysis, including the effects of the isocurvature perturbations, typically encoded in the transfer functions, is left to a more extensive future discussion.

The model of  $j$ -inflation is part of a class of functions comprised of genus zero modular functions. In the framework of modular inflation these provide a reservoir of different models that lend themselves to a similar analysis. It is furthermore possible to consider modular inflation based on more general forms of higher level by reducing the symmetry group to some congruence subgroup of level  $N$ .

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## References

- [1] A.D. Linde, *Chaotic inflation*, Phys. Lett. **B129** (1983) 177
- [2] K. Enqvist and J. Maalampi, *Problems with chaotic inflation* Phys. Lett. **B180** (1986) 14
- [3] K. Freese, J.A. Frieman and A.V. Olinto, *Natural inflation with pseudo-Nambu-Goldstone bosons*, Phys. Rev. Lett. **65** (1990) 3233; F.C Adams, J.R. Bond, K. Freese, J.A. Freeman and A.V. Olinto, *Natural inflation: particle physics models, power law spectra for large structure, and constraints from COBE*, Phys. Rev. **D47** (1993) 426, arXiv: hep-ph/9207245
- [4] E. Silverstein and A. Westphal, *Monodromy in the CMB: gravity waves and string inflation*, Phys. Rev. **D78** (2008) 106003, arXiv: 0803.3085 [hep-th]; L. McAllister, E. Silverstein and A. Westphal, *Gravity waves and linear inflation from axion monodromy*, Phys. Rev. **D82**, 046003 (2010) [arXiv: 0808.0706 [hep-th]]
- [5] E. Pajer and M. Peloso, *A review of axion inflation in the era of Planck*, Class. Quant. Grav. **30** (2013) 214002, [arXiv: 1305.3557 [hep-th]]
- [6] E. Silverstein, *Les Houches lectures on inflationary observables and string theory*, arXiv: 1311.2312 [hep-th]
- [7] P.A.R. Ade *et al.*, *Detection of B-mode polarization at degree angular scales by BICEP2*, Phys. Rev. Lett. **112** (2014) 241101, [arXiv: 1403.3985 [astro-ph.CO]]
- [8] R. Flauger, J.C. Hill and D.N. Spergel, *Toward an understanding of foreground emission in the BICEP2 region*, JCAP **08** (2014) 39, [arXiv: 1405.7351 [astro-ph.CO]]
- [9] M.J. Mortonson and U. Seljak, *A joint analysis of PLANCK and BICEP2 B-modes including dust polarization uncertainty*, JCAP **10** (2014) 35, [arXiv: 1405.5857 [astro-ph.CO]]
- [10] D.H. Lyth, *What would we learn by detecting a gravitational wave signal in the cosmic background anisotropy?* Phys. Rev. Lett. **78** (1997) 1861
- [11] E. Silverstein and D. Tong, *Scalar speed limits and cosmology: acceleration from deceleration*, Phys. Rev. **D70** (2004) 103505, [arXiv: hep-th/0310221]; M. Alishahiha,

- E. Silverstein and D. Tong, *DBI in the sky: non-Gaussianity from inflation with a speed limit*, Phys. Rev. **D70** (2004) 123505, [arXiv: hep-th/0404084]
- [12] D. Langlois, *Lectures on inflation and cosmological perturbations*, Lect. Notes Phys. **800** (2010) 1, [arXiv: 1001.5259 [astro-ph]]
- [13] A. Borel and W. Casselman, *Automorphic forms, representations, and L-functions*, Proc. Pure Math. **33**, Amer. Math. Soc., 1979
- [14] Harish-Chandra, *Automorphic forms on semi-simple Lie groups*, Springer 1968
- [15] R.P. Langlands, *On the functional equations satisfied by Eisenstein series*, LNM 544, Springer 1976
- [16] L. Kronecker, *Über die elliptische Functionen für welche complexe Multiplication stattfindet*, Monatsber. Königl. Preuss. Akad. der Wiss. zu Berlin (1857) 455 – 460; also in Werke, Vol. 2, 177 – 184
- [17] C. Hermite, *Sur la résolution de l'équation du cinquième degré*, C. R. Acad. Sci. Paris Sér. I Math (1858) 508
- [18] M. Sasaki and E. Stewart, *A general analytic formula for the spectral index of the density perturbations produced during inflation*, Prog. Theor. Phys. **95**, 71 (1996) [astro-ph/9507001]
- [19] A.A. Starobinsky, *Spectrum of relict gravitational radiation and the early state of the universe*, JETP Lett. **30** (1979) 682, [Pisma Zh. Eksp. Teor. Fiz. **30** (1979) 719 – 723]
- [20] P.A.R. Ade et al., *Planck 2013 results XXII. Constraints on inflation*, Astron. Astrophys. **571** (2014) A22, [arXiv: 1303.5082 [astro-ph.CO]]
- [21] D. Baumann, *CMBPol mission concept study: probing inflation with CMB polarization*, AIP Conf. Proc. **1141** (2009) 10, [arXiv: 0811.3919 [astro-ph.CO]]
- [22] P. Andre, *PRISM (Polarized radiation imaging and spectroscopy mission): An extended white paper*, JCAP **02** (2014) 006, [arXiv: 1310.1554 [astro-ph.CO]]
- [23] P.A.R. Ade et al., *Planck 2013 results XXIV, Constraints on primordial non-Gaussianity*, Astron. Astrophys. **574** (2014) A24, [arXiv: 1303.5084 [astro-ph.CO]]